Outline of the Proof of the Main Theorem

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The starting point of my approach is the notion of variation through solutions. In my monograph “The Action Principle and Partial Differential Equations”, which treats general systems of partial differential equations arising from an action principle, I showed that such “first order” variations are associated to a linearized Langangian, on the basis of which energy currents are constructed. It is through energy currents and their associated integral identities that the estimates, essential to the approach, are derived. Here we consider the first order variations which correspond to the one-parameter subgroups of the Poincaré group, the isometry group of Minkowski spacetime, extended by the one-parameter scaling or dilation group, which leave the surrounding constant state invariant. The higher order variations correspond to the one-parameter groups of diffeomorphisms generated by a set of vectorfields, the commutation fields.

The construction of an energy current requires a multiplier vectorfield which at each point belongs to the closure of the positive component of the inner characteristic core in the tangent space at that point. In the case of irrotational fluid mechanics the characteristic in the tangent space at a point consists only of the sound cone at that point and this requirement becomes the requirement that the multiplier vectorfield be non-spacelike and future directed with respect to the acoustical metric h.

I use two multiplier vectorfields. The first multiplier field is the vectorfield K_0:

$$K_0 = (\eta_0^{-1} + \alpha^{-1}\kappa)L + \underline{L}, \quad \underline{L} = \alpha^{-1}\kappa L + 2T \tag{1}$$

The second multiplier field is the vectorfield K_1 defined by:

$$K_1 = (\omega/\nu)L \tag{2}$$

Here ν is the mean curvature of the wave fronts $S_{t,u}$ relative to their null normal L. However ν is defined not relative to the acoustical metric $h_{\mu\nu}$ but rather relative to a conformally related metric $\tilde{h}_{\mu\nu}$:

$$\tilde{h}_{\mu\nu} = \Omega h_{\mu\nu} \tag{3}$$

It turns out that there is a choice of conformal factor Ω such that a first order variation $\dot{\phi}$ of the wave function ϕ satisfies the linear wave equation relative to the metric $\tilde{h}_{\mu\nu}$:

$$\Box_{\tilde{h}}\dot{\phi} = 0 \tag{4}$$

This choice defines Ω and the definition makes Ω the ratio of a function of σ to the value of this function in the surrounding constant state, thus Ω is equal to unity in the constant state. It turns out moreover that Ω is bounded above and below by positive constants. The function ω appearing in 3 is required to have linear growth in t and to be such that $\Box_{\tilde{h}}\omega$ is suitably bounded. To each variation ψ, of any order, there are energy currents associated to ψ and to K_0 and K_1 respectively.

These currents define the energies $\mathcal{E}_0^u[\psi](t)$, $\mathcal{E}_1'^u[\psi](t)$, and fluxes $\mathcal{F}_0^t[\psi](u)$, $\mathcal{F}_1'^t[\psi](u)$. For given t and u the energies are integrals over the exterior of the surface $S_{t,u}$ in the hyperplane Σ_t, while the fluxes are integrals over the part of the outgoing null hypersurface C_u between the hyperplanes Σ_0 and Σ_t. It is these energy and flux integrals, together with a spacetime integral $K[\psi](t,u)$ associated to K_1, to be discussed below, which are used to control the solution.

Evidently, the means by which the solution is controlled depend on the choice of the acoustical function u, the level sets of which are the outgoing null hypersurfaces C_u. The function u is determined by its restriction to the initial hyperplane Σ_0.

The divergence of the energy currents, which determines the growth of the energies and fluxes, itself depends on ${}^{(K_0)}\tilde{\pi}$, in the case of the energy current associated to K_0, and ${}^{(K_1)}\tilde{\pi}$, in the case of the energy current associated to K_1. Here for any vectorfield X in spacetime, I denote by ${}^{(X)}\tilde{\pi}$ the Lie derivative of the conformal acoustical metric $\tilde{h}$ with respect to X. I call ${}^{(X)}\tilde{\pi}$ the *deformation tensor* corresponding to X. In the case of higher order variations, the divergences of the energy currents depend also on the ${}^{(Y)}\tilde{\pi}$, for each of the commutation fields Y to be discussed below.

All these deformation tensors ultimately depend on the acoustical function u, or, what is the same, on the geometry of the foliation of spacetime by the outgoing null hypersurfaces C_u. Recall from the previous lecture that the most important geometric property of this foliation from the point of view of the study of shock formation is the density of the packing of its leaves C_u. One measure of this density is the inverse spatial density of the wave fronts, that is, the inverse density of the foliation of each spatial hyperplane Σ_t by the surfaces $S_{t,u}$. This is the function κ. Another measure is the inverse temporal density of the wave fronts, the function μ.

The two measures are related by:

$$\mu = \alpha\kappa \tag{5}$$

where α is the inverse density, with respect to the acoustical metric, of the foliation of spacetime by the hyperplanes Σ_t. The function α is expressed directly in terms of the 1-form $\beta = d\phi$. It turns out moreover, that it is bounded above and below by positive constants. Consequently μ and κ are equivalent measures of the density of the packing of the leaves of the foliation of spacetime by the C_u.

Recall from the previous lecture that shock formation is characterized by the blow up of this density or equivalently by the vanishing of κ or μ.

The other entity, besides κ or μ which describes the geometry of the foliation by the C_u is the second fundamental form of the C_u. Since the C_u are null hypersurfaces with respect to the acoustical metric h their tangent hyperplane at a point is the set of all vectors at that point which are h-orthogonal to the generator L, and L itself belongs to the tangent hyperplane, being h-orthogonal to itself. Thus the second fundamental form χ of C_u is intrinsic to C_u and in terms of the metric $\not{h}$ induced by the acoustical metric on the $S_{t,u}$ sections of C_u, it is given by:

$$\not{\mathcal{L}}_L \not{h} = 2\chi \tag{6}$$

where $\not{\mathcal{L}}_X \vartheta$ for a covariant $S_{t,u}$ tensorfield ϑ denotes the restriction of $\mathcal{L}_X \vartheta$ to $TS_{t,u}$.

The acoustical structure equations are:

The propagation equation for χ along the generators of C_u.

The Codazzi equation which expresses $\text{d}\!\!\!/\text{iv}\,\chi$, the divergence of χ intrinsic to $S_{t,u}$, in terms of $d\!\!\!/\text{tr}\chi$, the differential on $S_{t,u}$ of $\text{tr}\chi$, and a component of the acoustical curvature and of k, the second fundamental form of the Σ_t relative to h.

The Gauss equation which expresses the Gauss curvature of $(S_{t,u}, h\!\!\!/)$ in terms of χ and a component of the acoustical curvature and of k.

An equation which expresses $\mathcal{L}\!\!\!/_T\chi$ in terms of the Hessian of the restriction of μ to $S_{t,u}$ and another component of the acoustical curvature and of k.

These acoustical structure equations seem at first sight to contain terms which blow up as κ or μ tend to zero. The analysis of the acoustical curvature then shows that the terms which blow up as κ or μ tend to zero cancel.

The most important acoustical structure equation from the point of view of the formation of shocks is the propagation equation for μ along the generators of C_u:

$$L\mu = m + \mu e \tag{7}$$

where the function m given by:

$$m = \frac{1}{2}(\beta_L)^2 \left(\frac{dH}{d\sigma}\right)_s (T\sigma) \tag{8}$$

and the function e depends only on the derivatives of the β_α, the rectangular components of the 1-form $\beta = d\phi$, tangential to the C_u.

It is the function m which determines shock formation, when being negative, causing μ to decrease to zero.

I first establish a theorem, the fundamental energy estimate, which applies to a solution of the homogeneous wave equation in the acoustical spacetime, in particular to any first order variation. The proof of this theorem relies on certain bootstrap assumptions on the acoustical entities. The most crucial of these assumptions concern the behavior of the function μ.

To give an idea of the nature of these assumptions, one of the assumptions required to obtain the fundamental energy estimate up to time s is:

$$\mu^{-1}(T\mu)_+ \leq B_s(t) \quad : \text{ for all } t \in [0, s] \tag{9}$$

where $B_s(t)$ is a function such that:

$$\int_0^s (1+t)^{-2}[1+\log(1+t)]^4 B_s(t)dt \leq C \tag{10}$$

with C a constant independent of s. Here T is the vectorfield defined above and we denote by f_+ and f_-, respectively the positive and negative parts of an arbitrary function f.

This assumption is then established by a certain proposition with $B_s(t)$ the following function:

$$B_s(t) = C\sqrt{\delta_0}\frac{(1+\tau)}{\sqrt{\sigma-\tau}} + C\delta_0(1+\tau) \tag{11}$$

where $\tau = \log(1+t)$, $\sigma = \log(1+s)$, and δ_0 is a small positive constant appearing in the final bootstrap assumption.

The spacetime integral $K[\psi](t,u)$ mentioned above, is essentially the integral of

$$-\frac{1}{2}(\omega/\nu)(L\mu)_-|\not{d}\psi|^2$$

in the spacetime exterior to C_u and bounded by Σ_0 and Σ_t.

Another assumption states that there is a positive constant C independent of s such that in the region below Σ_s where $\mu < \eta_0/4$ we have:

$$L\mu \leq -C^{-1}(1+t)^{-1}[1+\log(1+t)]^{-1} \quad (12)$$

In view of this assumption, the integral
$K[\psi](t,u)$ gives effective control of the derivatives of the variations tangential to the wave fronts in the region where shocks are to form. The same assumption, which is then established by a certain proposition, also plays an essential role in the study of the singular boundary.

The final stage of the proof of of the fundamental energy estimate is the analysis of system of integral inequalities in two variables t and u satisfied by the five quantities $\mathcal{E}_0^u[\psi](t)$, $\mathcal{E}_1'^u[\psi](t)$, $\mathcal{F}_0^t[\psi](u)$, $\mathcal{F}_1'^t[\psi](u)$, and $K[\psi](t,u)$.

After this, the commutation fields Y, which generate the higher order variations, are defined. They are five: the vectorfield T which is tranversal to the C_u, the field $Q = (1+t)L$ along the generators of the C_u and the three rotation fields $R_i : i = 1, 2, 3$ which are tangential to the $S_{t,u}$ sections. The latter are defined to be $\Pi \overset{\circ}{R}_i : i = 1, 2, 3$, where the $\overset{\circ}{R}_i$ $i = 1, 2, 3$ are the generators of spatial rotations associated to the background Minkowskian structure, while Π is the h-orthogonal projection to the $S_{t,u}$.

Expressions for the deformation tensors ${}^{(T)}\tilde{\pi}$, ${}^{(Q)}\tilde{\pi}$, and ${}^{(R_i)}\tilde{\pi}$: $i = 1, 2, 3$ are then derived, which show that these depend on the acoustical entities μ and χ.

The higher order variations satisfy inhomogeneous wave equations in the acoustical spacetime, the source functions depending on the deformation tensors of the commutation fields. These source functions give rise to error integrals, that is to spacetime integrals of contributions to the divergence of the energy currents.

The expressions for the source functions and the associated error integrals show that the error integrals corresponding to the energies of the $n+1$st order variations contain the nth order derivatives of the deformation tensors, which in turn contain the nth order derivatives of χ and $n+1$st order derivatives of μ. Thus to achieve closure, we must obtain estimates for the latter in terms of the energies of up to the $n+1$st order variations. Now, the propagation equations for χ and μ give appropriate expressions for $\not{\mathcal{L}}_L\chi$ and $L\mu$. However, if these propagation equations, which may be thought of as ordinary differential equations along the generators of the C_u, are integrated with respect to t to obtain the acoustical entities χ and μ themselves, and their spatial derivatives are then taken, a loss of one degree of differentiability would result and closure would fail.

I overcome this difficulty in the case of χ by considering the propagation equation for $\mu\text{tr}\chi$. I show that, by virtue of a wave equation for σ, which follows from the wave equations satisfied by the first variations corresponding to the spacetime translations, the principal part on the right hand side of this propagation equation can be put into the form $-L\check{f}$ of a derivative of a function $-\check{f}$ with respect to L. This function is then brought to the left hand side and we obtain a propagation equation for $\mu\text{tr}\chi + \check{f}$. In this equation $\hat{\chi}$, the trace-free part of χ enters, but the propagation equation in question is considered in conjuction with the Codazzi equation, which constitutes an elliptic system on each $S_{t,u}$ for $\hat{\chi}$, given $\text{tr}\chi$. We thus have an ordinary differential equation along the generators of C_u coupled to an elliptic system on the $S_{t,u}$ sections.

More precisely, the propagation equation which is considered at the same level as the Codazzi equation is a propagation equation for the $S_{t,u}$ 1-form $\mu \not{d}\mathrm{tr}\chi + \not{d}\check{f}$, which is a consequence of the equation just discussed. To obtain estimates for the angular derivatives of χ of order l we similarly consider a propagation equation for the $S_{t,u}$ 1-form:

$$^{(i_1 \ldots i_l)}x_l = \mu \not{d}(R_{i_l} \ldots R_{i_1} \mathrm{tr}\chi) + \not{d}(R_{i_l} \ldots R_{i_1} \check{f})$$

In the case of μ the aforementioned difficulty is overcome by considering the propagation equation for $\mu \not\triangle \mu$, where $\not\triangle \mu$ is the Laplacian of the restriction of μ to the $S_{t,u}$. I show that by virtue of a wave equation for $T\sigma$, which is a consequence of the wave equation for σ, the principal part on the right hand side of this propagation equation can again be put into the form $L\check{f}'$ of a derivative of a function $\check{f}'$ with respect to L. This function is then likewise brought to the left hand side and we obtain a propagation equation for $\mu \not\triangle \mu - \check{f}'$. In this equation $\hat{\not{D}}^2 \mu$, the trace-free part of the Hessian of the restriction of μ to the $S_{t,u}$ enters, but the propagation equation in question is considered in conjuction with the elliptic equation on each $S_{t,u}$ for μ, which the specification of $\not\triangle \mu$ constitutes. Again we have an ordinary differential equation along the generators of C_u coupled to an elliptic equation on the $S_{t,u}$ sections.

To obtain estimates of the spatial derivatives of μ of order $l+2$ of which m are derivatives with respect to T we similarly consider a propagation equation for the function:

$$
\begin{aligned}
{}^{(i_1 \ldots i_{l-m})}x'_{m,l-m} = \mu R_{i_{l-m}} \ldots R_{i_1} (T)^m \not\Delta \mu \\
- R_{i_{l-m}} \ldots R_{i_1} (T)^m \check{\not{f}}'
\end{aligned}
$$

This allows us to obtain estimates for the top order spatial derivatives of μ of which at least two are angular derivatives. A remarkable fact is that the missing top order spatial derivatives do not enter the source functions, hence do not contribute to the error integrals.

The paradigm of an ordinary differential equation along the generators of a null hypersurface coupled to an elliptic system on the sections of the hypersurface was first encountered in my work with Sergiu Klainerman on the stability of Minkowski spacetime in general relativity.

Here, the appearence of the factor of μ, which vanishes where shocks originate, in front of

$$\not{d}R_{i_l}...R_{i_1}\mathrm{tr}\chi \text{ and } R_{i_{l-m}}...R_{i_1}(T)^m \not{\Delta}\mu$$

in the definitions of

$$^{(i_1...i_l)}x_l \text{ and } ^{(i_1...i_{l-m})}x'_{m,l-m}$$

above, makes the analysis quite delicate. This is compounded with the difficulty of the slow decay in time which the addition of the terms $-\not{d}R_{i_l}...R_{i_1}\check{f}$ and $R_{i_{l-m}}...R_{i_1}(T)^m\check{f}'$ forces.

The analysis requires a precise description of the behavior of μ itself, given by certain propositions, and a separate treatment of the condensation regions, where shocks are to form, from the rarefaction regions, the terms refering not to the fluid density but rather to the density of the stacking of the wave fronts. To overcome the difficulties the following weight function is introduced:

$$\overline{\mu}_{m,u}(t) = \min\left\{\frac{\mu_{m,u}(t)}{\eta_0}, 1\right\}, \quad \mu_{m,u}(t) = \min_{\Sigma_t^u} \mu \tag{13}$$

where Σ_t^u is the exterior of $S_{t,u}$ in Σ_t, and the quantities $\mathcal{E}_0^u[\psi](t)$, $\mathcal{E}_1'^u[\psi](t)$, $\mathcal{F}_0^t[\psi](u)$, $\mathcal{F}_1'^t[\psi](u)$, and $K[\psi](t,u)$ corresponding to the highest order variations are weighted with a power, $2a$, of this weight function.

The following lemma then plays a crucial role here as well as in the proof of the main theorem where everything comes together. Let:

$$M_u(t) = \max_{\Sigma_t^u} \left\{ -\mu^{-1}(L\mu)_- \right\},$$
$$I_{a,u} = \int_0^t \overline{\mu}_{m,u}^{-a}(t') M_u(t') dt' \tag{14}$$

Then under certain bootstrap assumptions in the past of Σ_s, for any constant $a \geq 2$, there is a positive constant C *independent of* s, u *and* a such that for all $t \in [0, s]$ we have:

$$I_{a,u}(t) \leq C a^{-1} \overline{\mu}_{m,u}^{-a}(t) \tag{15}$$

The acoustical assumptions on which the previous results depend are established, using the method of continuity, on the basis of the final bootstrap assumption, which consists only of pointwise estimates for the variations up to certain order.

The analysis of the structure of the terms containing the top order spatial derivatives of the acoustical entities shows that these terms can be expressed in terms of the 1-forms ${}^{(i_1 \dots i_l)}x_l$ and the functions ${}^{(i_1 \dots i_{l-m})}x'_{m,l-m}$. These contribute *borderline error integrals*, the treatment of which is the main source of difficulties in the problem. The borderline integrals are all proportional to the constant ℓ mentioned above, hence are absent in the case $\ell = 0$.

I should make clear here that the only variations which are considered up to this point are the variations arising from the first order variations corresponding to the group of spacetime translations. In particular the final bootstrap assumption involves only variations of this type, and each of the five quantities $\mathcal{E}^u_{0,[n]}(t)$, $\mathcal{F}^t_{0,[n]}(u)$, $\mathcal{E}'^u_{1,[n]}(t)$, $\mathcal{F}'^t_{1,[n]}(u)$, and $K_{[n]}(t,u)$, which together control the solution, is defined to be the sum of the corresponding quantity $\mathcal{E}^u_0[\psi](t)$, $\mathcal{F}^t_0[\psi](u)$, $\mathcal{E}'^u_1[\psi](t)$, $\mathcal{F}'^t_1[\psi](u)$, and $K[\psi](t,u)$, over all variations ψ of this type, up to order n.

To estimate the borderline integrals however, I introduce an additional assumption which concerns the first order variations corresponding to the scaling or dilation group and to the rotation group, and the second order variations arising from these by applying the commutation field T. This assumption is later established through energy estimates of order 4 arising from these first order variations and derived on the basis of the final bootstrap assumption, just before the recovery of the final bootstrap assumption itself. It turns out that the borderline integrals all contain the factor $T\psi_\alpha$, where $\psi_\alpha : \alpha = 0, 1, 2, 3$ are the first variations corresponding to spacetime translations and the additional assumption is used to obtain an estimate for $\sup_{\Sigma_t^u} \left(\mu^{-1} |T\psi_\alpha| \right)$ in terms of $\sup_{\Sigma_t^u} \left(\mu^{-1} |L\mu| \right)$, which involves on the right the factor $|\ell|^{-1}$.

Upon substituting this estimate in the borderline integrals, the factors involving ℓ cancel, and the integrals are estimated using the inequality 15. The above is an outline of the main steps in the estimation of the borderline integrals associated to the vectorfield K_0. The estimation of the borderline integrals associated to the vectorfield K_1, is however still more delicate. In this case I first perform an integration by parts on the outgoing null hypersurfaces C_u, obtaining hypersurface integrals over Σ_t^u and Σ_0^u and another spacetime volume integral. In this integration by parts the terms, including those of lower order, must be carefully chosen to obtain appropriate estimates, because here the long time behavior, as well as the behavior as μ tends to zero, is critical. Another integration by parts, this time on the surfaces $S_{t,u}$, is then performed to reduce these integrals to a form which can be estimated.

The estimates of the hypersurface integrals over Σ_t^u are the most delicate (the hypersurface integrals over Σ_0^u only involve the initial data) and require separate treatment of the condensation and rarefaction regions, in which the properties of the function μ, established by the previous propositions, all come into play.

In proceeding to derive the energy estimates of top order, $n = l + 2$, the power $2a$ of the weight $\overline{\mu}_{m,u}(t)$ is chosen suitably large to allow us to transfer the terms contributed by the borderline integrals to the left hand side of the inequalities resulting from the integral identities associated to the multiplier fields K_0 and K_1. The argument then proceeds along the lines of that of the fundamental energy estimate, but is more complex because here we are dealing with weighted quantities.

Once the top order energy estimates are established, I revisit the lower order energy estimates using at each order the energy estimates of the next order in estimating the error integrals contributed by the highest spatial derivatives of the acoustical entities at that order. I then establish a descent scheme, which yields, after finitely many steps, estimates for the five quantities $\mathcal{E}^u_{0,[n]}(t)$, $\mathcal{F}^t_{0,[n]}(u)$, $\mathcal{E}'^u_{1,[n]}(t)$, $\mathcal{F}'^t_{1,[n]}(u)$, and $K_{[n]}(t,u)$, for $n = l + 1 - [a]$, where $[a]$ is the integral part of a, in which weights no longer appear.

It is these unweighted estimates which are used to close the bootstrap argument by recovering the final bootstrap assumption. This is accomplished by the method of continuity through the use of the isoperimetric inequality on the wave fronts $S_{t,u}$, and leads to the main theorem.